Mid-term Exam | ISI Bangalore | September 2016

Differential Topology | 100 Points

Notes.
(a) Justify all your steps. You may use any result proved in class unless you have been asked to prove the same.
(b) $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers,

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1. [7 points] Verify that the intersection C of the two surfaces in $\mathbb{R}^3$ given by the locus of $x^2 - y = 0$ and $xy - z = 0$ is a submanifold of dimension 1 and give a parametrization of the tangent line to C at $p = (1, 1, 1)$.

2. [7 points] Prove that the dimension of a compact submanifold of $\mathbb{R}^N$ is strictly less than N.

3. [22 points]
   (i) Let $x_1, \ldots, x_N$ be the standard coordinate functions on $\mathbb{R}^N$ and let M be a k-dimensional submanifold of $\mathbb{R}^N$. Prove that for every $p \in M$, there exists a neighbourhood U of p and there exist k coordinates $x_{i_1}, \ldots, x_{i_k}$ such that the corresponding projection $f\colon U \to \mathbb{R}^k$ gives a diffeomorphism of U to an open subset V of $\mathbb{R}^k$. *(Hint: First solve the corresponding linear problem.)*
   (ii) Deduce that U in (i) is the graph of a smooth function $g = (g_{k+1}, \ldots, g_N)\colon V \to \mathbb{R}^{N-k}$.
   (iii) Prove that the locus of $y^3 - x^5$ in $\mathbb{R}^2$ is a not a submanifold around the origin.

4. [22 points] Let $f\colon X \to Y$ be a smooth map of manifolds.
   (i) If $\dim(X) = \dim(Y)$ and X is compact, prove that for any regular value $y \in Y$, the set $f^{-1}(y)$ is finite.
   (ii) Give an example where f is an immersion and $y \in Y$ is regular for f, but $f^{-1}(y)$ is infinite.
   (iii) If f is surjective, prove that $\dim(X) \geq \dim(Y)$.

5. [15 points] Define what it means for $f\colon X \to Y$ to be transversal to a submanifold $Z \subset Y$. Prove that in this case, $f^{-1}Z$ is a submanifold in X of codimension same as that of Z in Y.

6. [12 points] Recall that a manifold X is called *contractible* if its identity map is homotopic to some constant map $X \to \{x\}$ where x is a point of X. Prove that X is contractible if and only if for any arbitrary manifold Y, all the maps from Y to X are homotopic to each other.

7. [15 points]
   (i) Give an example of an uncountable dense subset of $[0, 1]$ having measure 0.
   (ii) Prove that if $X \subset \mathbb{R}^k$ and $Y \subset \mathbb{R}^l$ have measure 0, then so does $X \times Y \subset \mathbb{R}^{k+l}$.